INTRODUCTION

Most developers and publishers claim that their curricula are based on research, but few explicate their claims. In this chapter, we briefly assess the state of affairs regarding “research-based curricula” and present a model to mitigate weaknesses in the field that is based on coordinated interdisciplinary research ranging from cognitive science to scale-up. We describe an example in early mathematics.

Finding a curriculum that does not claim a research basis is difficult, but these claims are often vacuous, citing theories or empirical results vaguely (Clements, 2007, 2008; Clements & Sarama, 2013; Kinzie, Whittaker, McGuire, Lee, & Virginia, 2015). For example, they often cite research evidence relevant to the beginning or end of the curriculum development process. That is, at the beginning, “research-based” often indicates asserting that the curriculum was built upon broad theoretical frameworks or, with little specificity, “research on students’ thinking.” Such a research-to-practice model alone is inadequate, because it includes a one-way translation of research results to principles to instructional designs and therefore is often...
flawed in its presumptions, insensitive to changing goals in the content area, and unable to contribute to a revision of the theory and knowledge on which it is built (Clements, 2007).

At the other end, research validated may mean that effectiveness of the finished curriculum was evaluated. Not only does this leave out critical stages of a scientific research and development process (Battista & Clements, 2000; Clements & Battista, 2000; Doabler et al., 2014), but the research designs are often weak (Munter, Cobb, & Shekell, 2016). In the area of early mathematics, for example, of 78 elementary school programs evaluated, less than 10% had valid evidence of effectiveness and four of those had only “potentially positive” effects on achievement (Doabler et al., 2014). Why this state of affairs continues is explained by the confluence of many factors, such as instrumentalist views of mathematics and knowledge acquisition as simple transmission, a skepticism or rejection of mathematics curricula in the earliest years, lack of standards for curriculum development, a bias against design sciences, such as curriculum development in particular, in academe, and limited involvement of and communication between, relevant parties (for elaboration, see Clements & Sarama, 2013).

This is not to say that there have been no viable attempts to build valid research-based curricula. There are many (for lists of examples, see Clements, 2008; Day-Hess & Clements, 2017). However, they remain relatively small in number and frequently do not explicate the methods and findings of the development process.

To address these weaknesses, close the gap between research and practice, and increase the impact of research on the field (Cai et al., 2017), we need scientific approaches to the conceptualization, design, creation, implementation, and scale-up of curricula that are not just “based on” or “validated by” research but that were constructed, refined, and evaluated with a comprehensive program of research and development (Clements, 2007, 2008; Clements & Sarama, 2013).

THE CURRICULUM RESEARCH FRAMEWORK (CRF)

Based on a review of research and expert practice (Clements, 2008), we constructed and tested a framework for the construct of research-based curricula. The goal was to promote a valid scientific curriculum development program that addresses two basic questions—about effects and conditions—in three domains: practice, policy, and theory. For example, a curriculum development program should address not only the practical question of whether the curriculum is effective in helping children achieve specific learning goals, but also the conditions under which it is effective. Theoretically, the research program should also address why it is effective and why certain sets of conditions decrease or increase the curriculum’s effectiveness.
We developed the Curriculum Research Framework (CRF, Clements, 2007), which identifies three categories and ten phases of research and development, along with methods appropriate for each. A core feature of the CRF is that it is grounded in coordinated interdisciplinary research ranging from cognitive science to early childhood and mathematics education to implementation science to scale-up (the final scale-up phrase is complex and has its own elaborated model, Sarama & Clements, 2013).

Each phase must yield positive results to proceed to the next. This process can reveal weaknesses that have to be addressed and reevaluated (or the project halted, saving resources before large-scale evaluations are conducted, most likely yielding little to no benefits). This approach has higher validity than others for the same reason: Construct validity tests are more frequent and more trustworthy. For example, if research on students’ thinking and learning in the goal domain is not carefully reviewed or conducted, it is considerably less likely that later phases of development (curricula, professional development, implementation, etc.) will be successful.

The CRF and Early Mathematics

We first implemented the CRF in the field of early mathematics, given its importance (Clements & Sarama, 2014; Sarama & Clements, 2009) and the low use of mathematics curricula in the earliest years of schooling in the United States. For example, US teachers tend to use emerging curricula, whereas those in China use mathematics-specific curricula (Li, Chi, DeBey, & Baroody, 2015).

The CRF Enacted

As stated, the CRF includes ten phases for asserting that a curriculum is based on research, which can be ordered by the chronology of typical curriculum development, although they are cyclic or recursive (Clements, 2007, 2008; Clements & Sarama, 2013). In the remainder of this section, we briefly describe each phase and then illustrate how we enacted that phase in the Building Blocks research and development project, a NSF-funded early childhood mathematics research and curriculum development project that was the first to be based on the CRF.

Category I: A Priori Foundations

The first category is that of a priori foundations. Here, the nature of the phase is a focused version of the research-to-practice model. That is, the extant research is reviewed and implications for the nascent curriculum development effort drawn. The questions asked regard what is already known that can be
applied to an anticipated curriculum, concerning psychology, education, systemic change, and so forth in general (phase 1), the specific subject matter content, including the role it would play in students’ knowledge development (phase 2), and pedagogy, including the effectiveness of certain types of activities (phase 3).

A general guideline across these evaluation phases is that equity issues (Confrey & Lachance, 2000) be considered. For example, considerable thought should be given to the students who are envisioned as users and who participate in field tests; a convenience sample is often inappropriate. Systemic sociocultural issues should be considered as well (Tate, 1997). For Building Blocks, we used research on and conducted all field tests with two populations: Children from low-resource communities and children with special needs.

Phase 1. General A Priori Foundation

Developers review broad philosophies, theories, and empirical results on learning and teaching. Based on theory and research on early childhood learning and teaching (Clements & Sarama, 2007a), we determined that Building Blocks’ basic approach would be finding the mathematics in, and developing mathematics from, children’s activity. That is, we wanted to “mathematize” everyday activities, such as puzzles, songs, moving, and building. For example, teachers might help children mathematize moving their bodies in many ways. Children might count their steps as they walk. They might also move in patterns: step, step, hop; step, step, hop…. They might do both, counting as they walk, “one, two, three, four, five six, ….” These examples show that mathematizing means representing and elaborating everyday activities mathematically. Children create models of everyday situations with mathematical objects, such as numbers and shapes; mathematical actions, such as counting or transforming shapes; and their structural relationships—and use those understandings to solve problems. They learn to increasingly see the world through mathematical lenses.

Phase 2. Subject Matter A Priori Foundation

Developers review research and consult with experts to identify topics that make a substantive contribution to children’s mathematical development, are generative in children’s development of future mathematical understanding, and are interesting to children. We determined the topics that fit those criteria by considering what mathematics is culturally valued (e.g., standards from domain-specific organizations and states) and empirical research on what constituted the core ideas and skill areas of mathematics for young children (Clements, Sarama, & DiBiase, 2004). We then organized for the development of learning trajectories in the domains of number (subitizing, counting, sequencing, arithmetic), geometry (matching, naming, building, and combining shapes), patterning, and measurement.
Phase 3. Pedagogical A Priori Foundation

Developers review empirical findings on making activities educationally effective—motivating and efficacious—to create general guidelines for the generation of activities. As an example, research using computer software with young children (Clements & Swaminathan, 1995; Sarama & Clements, in press) showed that preschoolers can use computers effectively and that software can be made more effective by employing animation, children’s voices, and clear feedback. Although such software is only a small component of the Building Blocks curriculum, it makes a significant contribution, because research was used in its development, giving the developers information on how to make the software targeted and effective.

Another issue that should be considered is for whom the curriculum is intended (e.g., sophisticated reform-oriented teacher, reform-oriented reform teacher, traditional teacher). Is it intended to be “ahead of where the teacher is” or fit the teachers’ current practice (Martin A. Simon, personal communication, May 28, 2002)? We planned that Building Blocks would be considerably “ahead” of the teachers because most had little preparation in mathematics education. However, to reduce the unfamiliarity for them, we also connected all aspects of the curriculum to typical early childhood educational practice whenever this was consistent with (e.g., traditional scheduling) and especially when it strengthened (e.g., an emphasis on child development and processes, not just products) the mathematics.

Category II: Learning Model and Learning Trajectory

Within the second category is the most extensive and intensive development phase, in which developers’ structure activities in accordance with theoretically and empirically based models of children’s thinking. This phase involves the creation of research-based learning trajectories—One for every major topic. For this paper, we focus on just one of the many topics from the Building Blocks curriculum, subitizing, or the quick recognition of a number of items in a set without counting, from the Latin “to arrive suddenly.”

Phase 4. Structure According to Specific Learning Model and Learning Trajectory

The question is how the curriculum can be constructed to be consistent with, and build upon, students’ thinking and learning, which are posited to have characteristics and developmental courses that are not arbitrary, and therefore not equally amenable to various instructional approaches or curricular routes (this is based on our arching theory of hierarchic interactionalism, to which space constraints allow only short references, but see Sarama & Clements, 2009). What distinguishes phase 4 from phase 3, which concerns pedagogical
a prior foundations, is not only the focus on the child’s learning, rather than teaching strategies alone, but also on the iterative nature of its application.

Fig. 1 diagrams the influence of research that creates a chain of development around this core component of learning trajectories (Confrey, Gianopulos, McGowan, Shah, & Belcher, 2017). We first discuss the three rounded rectangles that represent three fields of research that support the development of each initial (or “Hypothetical”) Learning Trajectory.

Mathematical progressions contribute to the identification of goals. We posit that worthwhile goals are based on the big ideas of mathematics: those that are mathematically central and coherent, consistent with children’s thinking, and generative of future learning (i.e., they are part of a coherent mathematical progression, Clements & Conference Working Group, 2004; van Marle et al., 2018). The fundamental importance of cardinal understanding of whole numbers needs no justification and children’s first cardinal meanings for number words may be labels for small sets of subitized objects, even if they determined the labels by counting (Fuson, 1992b; Slusser & Sarnecka, 2011; Steffe, Thompson, & Richards, 1982). Subitizing introduces basic ideas of cardinality—”how many,” ideas of “more” and “less,” ideas of parts and wholes and their relationships, beginning arithmetic, and, in general, ideas of quantity. Developed well, these are related, forming webs of connected ideas that are the building blocks of mathematics through elementary, middle, and high school and beyond. Finally, the Common Core State Standards explicitly describes subitizing: “Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects” (National Governor’s Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 9).

Psychological research. Psychologically oriented research, from cognitive science, developmental psychology, education, and so forth, was particularly critical and extensive for the topic of subitizing. Indeed, we note that this research has shown that innate competencies that underlie subitizing are the first quantification mechanisms. For example, children as young as 6 months of age and probably younger are sensitive to number. They habituate to 1 vs. 2 or 3 objects (Antell & Keating, 1983; P. Starkey, Spelke, & Gelman, 1990). For example, shown repeated sets of 3, they eventually “get used to” the number, even as color, size, and arrangements change, and become more attentive only when a set with a different number such as 2 is shown. This indicates that infants are sensitive to small numerosities of a set of items before they are taught number words, counting, or finger patterns. Such research contributes, from a psychological perspective, to the importance of the goal of subitizing.

However, the main contribution of psychologically oriented research for all topics is to the creation of the developmental progression of the learning trajectories. Given the importance of this to the creation of a learning trajectory, we review the research in some detail to illustrate that creation
FIG. 1 The logic model for the Phase 4 of the Curriculum Research Framework.
Subitizing was initially defined in psychology (Kaufman, Lord, Reese, & Volkmann, 1949). Subitizing activity was differentiated from estimation as a unique form of visual number discrimination characterized by speed, exactness, and degree of confidence. For example, individuals identified sets of five or fewer objects quickly (<40ms/item in a perceptual field) in their recall times, accurately, and with confidence, and had higher accuracy rates than sets of more than five (Kaufman et al., 1949). Klahr found that subitizing did not rely on an encoding process, but in fact was an encoding process, explaining such different recall times when individuals subitized items between one and five. Agreeing that subitizing was a more “basic” skill than counting (Klahr & Wallace, 1976; Schaeffer, Eggleston, & Scott, 1974), Klahr (1973a) hypothesized that after items were encoded through subitizing activity, individuals stored matched patterned stimuli to numerical thinking structures in their long-term memory. Supporting this hypothesis, Fitzhugh (1978) found that some children could subitize sets of one or two but were not able to count them. None of these very young children, however, were able to count any sets that they could not subitize. Fitzhugh concluded that subitizing is a necessary precursor to counting. This research also began to define subitizing as supported by preattentional mechanisms (Klahr, 1973b; Trick & Pylyshyn, 1994) and a form of numerical encoding system (Klahr, 1973a).

However, others questioned whether subitizing is really a numerical process or rather a general sense of quantity. That is, some suggested that infants in “number” experiments may be responding to continuous quantities such as contour length, area, mass, or density rather than discrete number (Feigenson, Carey, & Spelke, 2002; Tan & Bryant, 2000). For example, infants dishabituated to changes in contour length when the number of objects was held constant, but they did not dishabituate to changes in number when contour length was held constant (Clearfield & Mix, 1999). There are, then, various empirical findings and theoretical models. Fig. 2 illustrates several. The “actions-on-objects” of each theory is a depiction of the mental actions, or processes, and what mental representation those processes act upon.

The scan paths view is that subitizing is the recognition of spatial patterns built through motion (Chi & Klahr, 1975; Glasersfeld, 1982; Klahr & Wallace, 1976). Numerical subitizing requires a subsequent reflective abstraction, which occurs when the child abstracts the mental from the sensory-motor contexts and is capable of reflecting on these actions.

The “accumulator” perspective views subitizing as a numerical process enabled by a mental functional equivalent of a number line that operates on both simultaneous and sequential items (cf. Huntley-Fenner, 2001). A pacemaker emits equivalent pulses at a constant rate. When a unitized item (e.g., a block taken as “one”) is encountered, a pulse is allowed to pass through a
gate, entering an accumulator—metaphorically like a squirt of water entering a tall glass. The gradations on the accumulator estimate the number in the collection of units, similar to height indicating the numerosity of squirts in the glass (Meck & Church, 1983). This model does not require that the accumulator has an exact representation of number (see also Feigenson, Dehaene, & Spelke, 2004), consistent with research indicating that children younger than 3 years tend not to represent any numbers except 1 and 2 precisely (Antell & Keating, 1983; Baroody, Lai, & Mix, 2005; Feigenson, Carey, & Hauser, 2002; Mix, Huttenlocher, & Levine, 2002).

The next theory in Fig. 2 holds that humans create “object files” that store data on each object’s properties and then can use these object files to respond to various situations. Thus some situations can be addressed by using the objects’ individuation or separateness as objects, and others can be addressed by using the analog properties of these objects, such as contour length (Feigenson, Carey, & Spelke, 2002). For example, children might use parallel-processed individuation for very small collections, but continuous extent when storage for individuation is exceeded.

The mental models view postulates that children represent numbers nonverbally and approximately, then nonverbally but exactly, and eventually via verbal, counting-based processes (Huttenlocher, Jordan, & Levine, 1994; Mix et al., 2002). Children cannot initially differentiate between discrete and continuous quantities, but represent both approximately using one or more perceptual cues such as contour length (Mix et al., 2002). Children gradually develop the ability to individuate objects, providing the ability to build notions of discrete number. About the age of 2 years, they develop representational, or symbolic, competence, allowing them to create mental models of collections, which they can retain, manipulate (move), add to or subtract from, and so forth (although the model does not adequately describe how cardinality is ultimately cognized and how comparisons are made). The symbolization differentiates this view from the related “object files” theory. Early nonverbal
capabilities then provide a basis for the development of verbally based numerical and arithmetic knowledge (young children are more successful on non-verbal than verbal versions of number and arithmetic tasks, Huttenlocher et al., 1994; Jordan, Hanich, & Uberti, 2003; Jordan, Huttenlocher, & Levine, 1992; Jordan, Huttenlocher, & Levine, 1994; Levine, Jordan, & Huttenlocher, 1992).

The abstract model postulates an evolutionarily based module—a distinct mental component that is dedicated to a particular process or task and is unavailable for general processing. A number perception module would process number-based information directly, but only generates approximations of the quantity for quantities above the subitizing range (Dehaene, 1997). This approximate number system is thought to guide the development of children’s initial understanding of cardinality (van Marle et al., 2018) and eventually contributes to the development of whole number counting. Researchers have used findings from both humans and nonhuman animals to support this position (Gallistel & Gelman, 2005).

For our learning model, we created a synthesis of these positions. What infants quantify are collections of rigid objects; that is, they do not quantify sequences of sounds and events or materials that are nonrigid and noncohesive such as water (Huntley-Fenner, Carey, & Solimando, 2002). Such quantifications, including number, begin as an undifferentiated, innate notion of “amount.” Object individuation, which occurs early in preattentive processing (and is a general, not numerical, process—we are not arguing that all subitizing is preattentive, cf. Moore & Ashcraft, 2015), helps lay the groundwork for differentiating discrete from continuous quantity. Multiple systems are employed, including an object file system that stores information about the objects, some or all of which is used depending on the situation, and an estimator (accumulator) mechanism that stores analog quantitative information only (Feigenson, Carey, & Spelke, 2002; Gordon, 2004; Johnson-Pynn, Ready, & Beran, 2005). This estimator may also include a set of number filters (a cognitive scheme that detects the numerosity of a small group), each tuned to an approximate number of objects (e.g., 2), although they overlap (Nieder, Freedman, & Miller, 2002). The child encountering small sets opens object files for each in parallel. If the situation elicits quantitative comparisons, continuous extent is usually retrieved and used. For example, by about a half-year of age, infants may represent very small numbers (1 or 2) as individuated objects (close to the “mental models” view). However, large numbers in which continuous extent varies or is otherwise not reliable (McCrink & Wynn, 2004) may be processed by the analog estimator as a collection of binary impulses (as are event sequences later in development, see the “analog” column of Fig. 2), but not by exact enumeration (Shuman & Spelke, 2005) by a brain region that processes quantity (size and number, undifferentiated, Pinel, Piazza, Le Bihan, & Dehaene, 2004). Without language support, these are inaccurate processes for numbers above two (Gordon, 2004).
To compare quantities, correspondences are processed. Initially, these are inexact estimates comparing the results of two estimators, depending on the ratio between the sets (Johnson-Pynn et al., 2005). Once the child can represent objects mentally, they can also make exact correspondences between these nonverbal representations, and eventually develop a quantitative notion of that comparison (e.g., ‘not just that ●●● is more than ●●’, but also that it contains one more ●).

Even these correspondences, however, do not necessarily imply a cardinal representation of the collection (a representation of the collection as a specific numerosity). That is, we still must distinguish between noncardinal representations of a collection and explicit cardinal representations. Indeed, a neuroimaging study found that brain regions that represent numerical magnitude also represent spatial magnitude, such as the relations between sizes of objects, and thus may not be numerical in function (Pinel et al., 2004). Only for the latter does the individual apply an integration operation (Steffe & Cobb, 1988) to create a composite with some numerical index. This integration operation uses present cognitive schemes to project and reorganize actions so they are considered mathematical objects. Some claim that the accumulator yields a cardinal output; however, it may be quantitative, and even numerical in some situations, and still—because it indexes a collection using an abstract, cross-modality system for numerical magnitude (cf. Lourenco & Longo, 2011; Shuman & Spelke, 2005)—it may lack an explicit cardinality. This system would not necessarily differentiate between ordinal and cardinal interpretations. Comparisons, such as correspondence mapping, might still be performed, but only at an implicit level (cf. Sandhofer & Smith, 1999). (It is possible to index a numerical label without attributing explicit cardinality. For example, lower animal species seem to have some perceptual number abilities, but only birds and primates also have shown the ability to connect a perceived quantity with a written mark or auditory label, Davis & Perusse, 1988.) In this view, only with experience representing and naming collections is an explicit cardinal representation created.

In summary, early quantitative abilities exist, but they may not initially constitute systems that can be said to have an explicit number concept. Instead, they may be premathematical, foundational abilities (cf. Clements et al., 2004) that develop and integrate slowly, in a piecemeal fashion (Baroody, Benson, & Lai, 2003). The explicit, cultural, numeral-based sense of number develops in interaction with, but does not replace (indeed, may always be based on, Gallistel & Gelman, 2005) the analog sense of number.

We next turn toward the cognitive development of subitizing. A main point is that it does develop. In contrast to what might be expected from a nativist perspective, many innate systems such as subitizing develop based on experience. Perceptual subitizing (Clements, 1999; see also theoretical justification in Karmiloff-Smith, 1992) is closest to the original definition of subitizing: Recognizing a number without consciously using other mental or mathematical
processes and then naming it. Thus perceptual subitizing employs a preattentional, encoding quantitative process but adds an intentional numerical process; that is, infant sensitivity to number is not (yet) perceptual subitizing. The term “perceptual” applies only to the quantification mechanism as phenomenologically experienced by the person; the intentional numerical labeling, of course, makes the cognitive act conceptual.

A main aspect of the developmental progression for subitizing is, of course, the size of the numbers. As we have seen, children first perceptually subitize sets of only one or two accurately. Indeed, in laboratory research, children can accurately differentiate one from “more than one” at about 33 months of age (Wynn, 1992). Between 35 and 37 months, they differentiate between one and two, but not larger numbers. A few months later, at 38–40 months, they identify three as well. After about 42 months, they can subitize all numbers that they can count, four and higher, at about the same time. However, research in natural, child-initiated settings shows that the development of these abilities can occur much earlier, with children working on one and two around their second birthdays or earlier (Mix, Sandhofer, & Baroody, 2005). Further, some children may begin with “two” rather than “one” (Spelke, 2017). These studies suggest that language and social interactions interact with internal factors in development, as well as showing that number knowledge develops in levels, over time (see also Gordon, 2004). Most studies suggest that children begin recognizing “one,” then “one” and “two,” then “three” and then “four,” whereupon they learn to count and know other numbers (Le Corre, Van de Walle, Brannon, & Carey, 2006).

Complementary research from the field of mathematics education has documented that most kindergartners appear to have good competence recognizing two and three, with most recognizing four and some recognizing higher numbers (note that different tasks were used, some of which did not limit time, which we call “recognition of number” rather than subitizing, so wide ranges are expected). A recent study of low-income children beginning pre-K, using a short-exposure subitizing task, report 2%–14% accuracy for 3, 0%–5% for 4, and virtually no competence with 5, 8, or 10 (Sarama & Clements, 2011). Thus children appear to be most confident with very small numbers, but those from less advantaged environments may not achieve the same skills levels as their more advantaged peers. Some special populations find subitizing particularly difficult. Only a minority (31%) of children with moderate mental handicaps (chronological ages 6–14 years) and a slight majority (59%) of children with mild mental handicaps (ages 6–13) successfully subitize sets of three and four (Baroody, 1986; see also Butterworth, 2010). Some children with learning disabilities could not subitize even at 10 years of age (Chu, van Marle, & Geary, 2013; Geary, Hoard, & Bailey, 2012; Koontz & Berch, 1996). Early deficits in spatial pattern recognition may be the source of difficulty (Ashkenazi, Mark-Zigdon, & Henik, 2013). Subitizing in preschool is a better predictor of later mathematics success for children with ASD (autism spectrum
disorder) than for typically developing children (Titeca, Roeyers, Josephy, Ceulemans, & Desoete, 2014).

A second type of subitizing, conceptual subitizing (Clements, 1999), plays an advanced organizing role, as seeing “10” on a pair of dice by recognizing the two collections (via perceptual subitizing) and composing them as units of units (Steffe & Cobb, 1988). There is empirical evidence for this distinction (see also the later term groupitizing in G. S. Starkey & McCandliss, 2014; Trick & Pylyshyn, 1994). Some research suggests that only the smallest numbers, perhaps up to three, are actually perceptually recognized; thus sets of 1–3 may be perceptually recognized, sets of 3 to about 6 may be composed (i.e., subitized sets are combined) with the person not necessarily aware of the process. Conceptual subitizing as we use the term refers to recognition in which the person uses partitioning strategies explicitly. Composing and decomposing are combining and separating operations that help children develop generalized part-whole relations, one of the most important accomplishments in arithmetic (National Research Council, 2001).

MacDonald and Wilkins (2016) elaborated on types of subitizing that may serve as a transition from perceptual to conceptual subitizing, in which children justified a total number (such as five) “because I saw three and two,” but without and explicit use of the subgroups nor a recognition of the part-whole relationships involved. They also identified two types of conceptual subitizing. In rigid conceptual subitizing, children may subitize “two, two, and one” each time they are shown a wide variety of “five.” They then develop flexible conceptual subitizing, as they build different ways of composing items (e.g., two and three; four and one).

The spatial arrangement of sets also influences how difficult they are to subitize (Mandler & Shebo, 1982). People from the primary grades to college usually find rectangular arrangements easiest, followed by linear, circular, and scrambled arrangements (Beckwith & Restle, 1966; Wang, Resnick, & Boozer, 1971). The only change across these ages is rectangular arrangements were much faster for the oldest students, who could multiply. Certain arrangements are easier for specific numbers. Arrangements yielding a better “fit” for a given number are easier (Brownell, 1928). Children make fewer errors for ten dots than for eight with the “domino five” arrangement, but fewer errors for eight dots for the “domino four” arrangement.

For young children, however, neither of these arrangements is easier for any number of dots. Indeed, children 2–4 years old show no differences between any arrangements of four or fewer items (Potter & Levy, 1968). For larger numbers, the linear arrangements are easier than rectangular arrangements. It may be that many preschool children do not use decomposing (conceptual subitizing). They can learn to conceptually subitize, though older research suggested that first graders’ limit for subitizing scrambled arrangements is about four or five (Dawson, 1953). If the arrangement does not lend itself to grouping, people of any age have more difficulty with larger
sets (Brownell, 1928). They also take more time with larger sets (Beckwith & Restle, 1966).

Another categorization involves the different types of things people can subitize. Spatial patterns such as those on dice are just one type. Other patterned modalities include finger and rhythmic patterns.

Fig. 3 presents the subitizing trajectory; here we focus on the first three columns that document the developmental progression, including the usual age of acquisition (first column; importantly—probably without effective interventions), and descriptions and illustrations (second column) of each of the levels of thinking, and the mental actions on objects we hypothesize account for each level of thinking.

Educationally oriented research contributes to the third component of learning trajectories, provision of instructional experiences. Regardless of the precise mental processes in the earliest years, subitizing appears to be phenomenologically distinct from counting and other means of quantification and deserves differentiated educational consideration. Supporting this assertion, there is little or no relationship between children’s performance on counting and subitizing tasks (Pepper & Hunting, 1998) and educational interventions have been shown to be effective (Clements & Sarama, 2008, 2013; Hannula, 2005), the topic to which we now turn.

What kinds of educational experiences are most effective? We first consider informal settings and strategies. Frequent naming of the numbers in groups appears essential. That is, the human facility with language links relations between different representations and thus making early premathematical cognition numeric (Gordon, 2004; Wiese, 2003). Even early grammatical structures for plurals (“dogs”) and quantifiers (“some,” “all,” and “a”) provide a framework that allows quantitative ideas to develop (Carey, 2004). More directly, experience naming collections helps children create explicit cardinal representations. This is a prolonged process. Children may initially make word-word mappings between requests for counting or numbers (e.g., “how many?”) to number words only after they have learned several number words (Sandhofer & Smith, 1999). Then they label some (small number) cardinal situations with the corresponding number word, that is, map the number word to the numerosity property of the collection. They begin this phase even before 2 years of age, but for some time, this applies mainly to the word “two,” a bit less to “one,” and with considerable less frequency, “three” and “four” (Fuson, 1992a; Wagner & Walters, 1982). Thus meaningful learning of number words may cause the transition to exact numerical representations (Baroody et al., 2005). This may provide the basis for understanding cardinality and other counting principles, as well as arithmetic ideas (Geary et al., 2012, 2018).

Such experiences promote not only subitizing competence, but motivation and habit of applying it. Such a tendency to spontaneously focus on numerosity is a distinct, mathematically significant process (Hannula, 2005;
The *goal* connects learning trajectories to the important *big ideas* in mathematics. The relevant big idea is that numbers can be used to tell how many, describe order, and measure; they involve numerous relations and can be represented in various ways. The specific goals include understanding that subitizing can be used to tell how many (perceptual subitizing), that a quantity can consist of parts and can be broken apart (decomposed) into the parts, and that those parts can be combined (composed) to form the whole (conceptual subitizing). Specific objectives are for children to achieve those understandings and fluency in perceptual and conceptual subitizing.

<table>
<thead>
<tr>
<th>Age* (yrs)</th>
<th>Developmental progression</th>
<th>Cognitive actions on objects</th>
<th>Sample instructional tasks</th>
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<tbody>
<tr>
<td>0–1</td>
<td><strong>Preexplicit number.</strong> Within the first year, the child is sensitive to small numbers, but does not have explicit, intentional knowledge of number.</td>
<td>An initial bootstrap: Implicit sensitivity to quantity with perceptual input. An object file system stores information about the objects (some or all of which is used depending on the situation). An estimator (accumulator-type) mechanism stores analog quantitative information. If the situation elicits quantitative comparisons, continuous extent is retrieved and used except in extreme circumstances (e.g., 1 and maybe 2 are processed as individuated objects; numbers in which continuous extent varies or is otherwise not reliable are processed by the analog estimator as a collection of binary impulses. To compare, bootstrap processes make a correspondence between estimators.</td>
<td><strong>Noticing collections.</strong> Provide a rich sensory environment, use words such as <em>more,</em> and use actions of adding objects.</td>
</tr>
<tr>
<td>1–2</td>
<td><strong>Small-collection namer.</strong> Names groups of 1 to 2, sometimes 3. Shown a pair of shoes, says, “Two shoes.”</td>
<td>Mental schemes (number filters) act on perceptions of collections of 1–3. Eventually, mental, nonverbal representations are developed of each object in such collections. The become</td>
<td><strong>Board games—small numbers.</strong> Play board games with a special die (number cube) or spinner that shows only 1, 2, and 3 dots (then add 0 to it).</td>
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<td>3</td>
<td><strong>Maker of small collections</strong>&lt;br&gt;Nonverbally makes a small collection (no more than 4, usually 1–3) with the same number as another collection via mental model (i.e., not necessarily by matching). Might also be verbal. When shown a collection of 3, makes another collection of 3.</td>
<td>Mental representations can be maintained and direct physical actions so that one perceived object corresponds to each represented object.</td>
<td><em>Get the number.</em> Ask children to get the right number of crackers or some other item for a small number of children.</td>
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<td>4</td>
<td><strong>Perceptual subitizer to 4.</strong>&lt;br&gt;Instantly recognizes collections up to 4 briefly shown, and verbally names the number of items.&lt;br&gt;When shown objects briefly, says, “Four.”</td>
<td>Schemes act on perceptual input (including collections, but also sounds, etc.) to identify sets of 0 to 4 (schemes may use lower-level schemes for 1-3 and combine them to recognize 4), each of which is associated with the verbal number name.</td>
<td><em>Snapshots.</em> Play &quot;Snapshots&quot; with collections of one to four objects, arranged in line or other simple arrangement, asking children to respond verbally with the number name. Start with the smaller quantities and easier arrangements, moving to those of moderate difficulty only when children are fully competent and confident.</td>
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<td>5</td>
<td><strong>Perceptual subitizer to 5.</strong>&lt;br&gt;Instantly recognizes briefly shown collections up to 5 items</td>
<td>Schemes act on perceptual input (including collections, but also sounds, etc.) to identify sets of 0 to 5 (schemes may use dot cards, starting with easy arrangements,</td>
<td><em>Snapshots.</em> Play &quot;Snapshots&quot; with dot cards, starting with easy arrangements,</td>
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<td>Conceptual subitizer to 5</td>
<td>Conceptual subitizer to 10</td>
<td>Conceptual subitizer to 20</td>
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<td>Verbally labels all arrangements to about 5, when shown only briefly. Asked “Why?” says, “Because I saw 3 and 2, and so I said 5.”</td>
<td>Verbally labels most briefly shown arrangements to 6, then up to 10, using groups. “In my mind, I made two groups of 3 and one more, so 7.”</td>
<td>Verbally labels structured arrangements up to 20, shown only briefly, using groups. “I saw three 5s, so 5, 10, 15.”</td>
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<td>An executive process determines whether an existing scheme can quantify the perceptual input; if so, that is used. If not, gestalt visual principals are used to partition the collection to identify two or more sets that existing schemes can quantify; the results of these are combined with pattern matching to known compositions.</td>
<td>As in the previous level, with the addition of other compositions.</td>
<td>As in the previous level, with the addition of other compositions and explicit knowledge of teens as ten and some number more.</td>
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<td><strong>Snapshots.</strong> Use different arrangements of the various modifications of “Snapshots” to develop conceptual subitizing and ideas about addition and subtraction. The goal is to encourage children to see the addends and the sum.</td>
<td><strong>Snapshots.</strong> Play “Snapshots” with larger quantities to develop ideas about addition and subtraction.</td>
<td><strong>Ten frame addition snapshots:</strong> Briefly show 2 ten frames to help children visualize addition combinations.</td>
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<td>Level</td>
<td>Description</td>
<td>Example</td>
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<td>7</td>
<td>Conceptual subitizer with place value and skip counting. Verbally labels structured arrangements, shown only briefly, using groups, skip counting, and place value. “I saw groups of 10s and 2s, so 10, 20, 30, 40, 42, 44, . . . 44!”</td>
<td>![ten frame example]</td>
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<td>As in the previous level, with the addition of other compositions and explicit place value knowledge.</td>
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<td>8</td>
<td>Conceptual subitizer with place value and multiplication. Verbally labels structured arrangements shown only briefly, using groups, multiplication, and place value. “I saw groups of 10s and 3s, so I thought, 3 tens is 30 and 4 threes is 12, so 42 in all.”</td>
<td>![ten frame example]</td>
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<td>As in the previous level, with the addition of other compositions and explicit knowledge of multiplication and place value.</td>
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<td>Snapshots with dots. Play “Snapshots” with structured groups that support the use of increasingly sophisticated mental strategies and operations.</td>
<td>![example]</td>
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*The ages are estimates based on extant research and are provided only as a general guide—age levels are strongly dependent on experience.

FIG. 3  A learning trajectory for recognition of number and subitizing.
Hannula et al., 2010). Children’s failure to focus on numerosity is not due to their lack of competences related to task demands (Lehtinen & Hannula, 2006). They simply have not developed the habit of focusing on numerosity, which is why it is important that all caregivers use number words throughout the day (LeFevre, Fast, et al., 2009; LeFevre, Skwarchuk, et al., 2009).

Turning to more formal school contexts, our theory of hierarchic interactionalism posits that “Instruction based on learning consistent with natural developmental progressions is more effective, efficient, and generative for the child than learning that does not follow these paths” (Sarama & Clements, 2009, p. 24). Therefore our learning trajectory moves from small numbers to greater numbers, but also from tasks that encourage mastery of perceptual subitizing before conceptual subitizing. Within each broad level, tasks following the research on types of objects and spatial arrangements of those objects (see the last column in Fig. 3).

To elaborate, because the goal is for the child to perceive the group as a quantity, we discourage the use of perceptually distinct objects (some teachers say, “I will use a yellow, a blue, and a red counter to help them remember”). We instead promote clear object-background contrast with identical objects (e.g., block circles on white background).

For arrangements, we follow the research described previously: With preschoolers, we begin with linear arrangements up to four or five, then move to rectangular arrangements, as well as dice and domino patterns.

Transitions to larger numbers are carefully planned. For instance, Logan and Zbrodoff (2003) found that the space between these groups of “twos” and “threes” afforded individuals more effective subitizing of four or more items (cf. Gebuis & Reynvoet, 2011), providing the transitional arrangements discovered by MacDonald and Wilkins (2016).

To continue to promote conceptual subitizing, we present arrangements that separate small sets, such as two and two separated symmetrically to see the set as “four” but also “two and two.” Symmetrical orientations appear to “free” children’s working memory resources. Individuals’ subitizing activity also has been found to be affected by the space between the items (Gebuis & Reynvoet, 2011); increasing this space may support young children (MacDonald, 2015; MacDonald & Shumway, 2016; MacDonald & Wilkins, 2016, 2017). We present many arrangements of each number that suggest different partitions of that number and move to extensive use of “five and ten frames.”

Computer technology can also contribute, especially software that moves children forward (or backward) along the learning trajectory automatically based on children’s performance (Clements & Sarama, 2007/2016). Tasks can be designed to fit the trajectory precisely. Errors are immediately detected and feedback given (Fig. 3; see other examples a learning trajectories.org). In these ways, work with computers can provide a unique and substantial contribution to children’s learning. Teaching and learning also should consider
a wide variety of social and contextual aspects (Aguirre et al., 2017; Confrey & Lachance, 2000; Secada, 1992; Tate, 1997).

Formative evaluation and validation of the learning trajectories (see Fig. 1) involves multiple methodologies, including consistency with extant research as just presented, statistical tests such as Rasch modeling (Clements, Sarama, & Liu, 2008; Sarama & Clements, 2011), and teaching experiments (Clements & Sarama, 1999; Sarama & Clements, 2004). All these were used in validating the learning trajectory in Fig. 3. Of course, formative evaluation of each learning trajectory continues through the following four phases.

Category III: Evaluation

The third category, evaluation, includes phases in which empirical evidence is collected to evaluate the learning trajectories and the synthesis of them into the complete curriculum. The objective is to evaluate the appeal, usability, and effectiveness of instantiations of the curriculum. Phase 5 focuses on questions of marketability. Phases 6–8 involve formative research, asking whether the curriculum is usable by, and effective with, teachers and students in expanding social contexts (with teachers familiar, and then new, to the materials), and, especially, how the curriculum can be improved.

Phase 5. Market Research

Market research is usually considered commercially oriented research about the customer, what the customer wants, and what they will buy. Typically, prototype materials are presented to “focus groups.” Publishers’ names and the results are hidden. Instead, we reveal all interviews and surveys we give to teachers, including their reaction to the subitizing learning trajectory—especially the activities.

The following three phases are also forms of formative evaluation. In contrast to market research, these phases often involve repeated test-and-revise cycles.

Phase 6. Formative Research: Small Group

Pilot testing with individuals or small groups of students is to be conducted on curricular components (e.g., a particular activity or software environment) or on sequences such as multiple subitizing activities (e.g., sequences of contiguous levels from Fig. 3). Early interpretive work evaluates components using a mix of model (or hypothesis) testing and model generation strategies, including clinical interviews, teaching experiences, and microethnographic approaches. The objective is to understand the meaning that students give to the curriculum objects and activities.

Evaluation focuses on consonance between the actions of the students and the learning model. If there are discrepancies, either the mental model, or the
way in which this model is instantiated in the curriculum, should be altered. In all cases, are students’ actions-on-objects enactments of their cognitive operations in the way the model posits are the focus. For example, do children engage in the activity Snapshots at the level of Conceptual Subitizer to 5 verbally label the whole and then the parts in ways the arrangements suggested? We found that largely so, although we found it useful if teachers encouraged children to name “different ways you could see it.” Are children who experience activities at level \( n \) more prepared to engage with tasks at level \( n + 1 \)? This we found to be consistent with observations of children’s behaviors; further, although some children showed signs of developing two levels during the same period of time, few “skipped” more than two. Thus developers use the cognitive model and learning trajectories as guides, and the software and activities as catalysts, to create more refined models of particular students or groups of students. Simultaneously, the developer describes what elements of the teaching and learning environment were observed as having contributed to, or hindered, student learning. As an example for subitizing, the Snapshots activity was written to have teachers secretly hide a number of objects under a cloth. We found two problems: (a) Teachers found it took too long to prepare each set, and (b) more important, we observed many teachers placing the objects one at a time—and children would count. This harmed their subitzing. Both problems were solved by switching to having patterns premade (with black circle stampers) on large numbers of paper plates.

Often this is the most iterative research-design phase; sometimes evaluation and redesign may cycle in quick succession, within a week to prepare modifications for another classroom, and sometimes as much as every 24 hours. Activities may be completed reconstituted, with edited or newly created ones tried the next day.

With so many research and development processes happening, and so many possibilities, extensive documentation is required. Documentation must allow researchers to relate findings to specific components and characteristics of the curriculum. Video or audio recordings (for later microgenetic analysis) and field notes are collected. This documentation also helps evaluate those components of the design that were based on intuition, esthetics, and subconscious beliefs.

**Phase 7. Formative Research: Single Classroom**

Although teachers are ideally involved in all phases, a special emphasis here is the process of curricular enactment. For example, a goal of the curriculum may be to help teachers interpret students’ thinking about the activities and the content they are designed to teach; support teachers’ learning of that content, especially which is probably new to teachers; and provide guidance regarding the external representation of content that the materials use. So, classroom-based teaching experiments help track and evaluate student learning, with the goal of
making sense of the curricular activities as they are experienced by individual students. At the same time, the class is observed for information concerning the usability and effectiveness of the curriculum. Ethnographic participant observation is used heavily because we wish to research the teacher and students as they interact. Thus the focus is on how the materials are used and how the teacher guides students through the activities. This phase often involves teachers working closely with the developers. That is, the class may be taught either by a team including one of the developers and the teacher, or by a teacher familiar with and intensively involved in curricula development. The goal is to examine learning in the context of the curriculum with teachers who can enact it with a high fidelity of implementation, as opposed to ascertaining how the curriculum works in classrooms in general, which is one focus of the next phase.

As one simple example for subitizing, we observed that although children were recognizing visual arrangements adequately, they seemed unable to transfer this ability to other activities (e.g., in a patterning activity in which children repeatedly “stomp, stomp, stomp, jump… stomp, stomp, stomp, jump…” not all children recognized the number of stomps). Therefore we mixed in temporal and kinesthetic subitizing tasks as soon as children gained competence with the visual presentations of a given number. This is significant because creating and using these patterns through conceptual subitizing helps children develop abstract number and arithmetic strategies. For example, children use temporal patterns when counting on. “I knew there were three more so I just said, nine … ten, eleven, twelve” (rhythmically gesturing three times, one “beat” with each count). They use finger patterns to figure out addition problems. For example, for 3 + 2, a child might put up a finger pattern they know as three, then put up two more (rhythmically—up, up) and then recognize the resulting finger pattern as “five.” Children who cannot subitize are handicapped in learning such arithmetic processes (Butterworth, 2010; Hannula et al., 2010).

**Phase 8. Formative Research: Multiple Classrooms**

In several classrooms, the entire class is observed for information about the effectiveness and usability of the curriculum, but more emphasis is placed on the usability by such teachers. Innovative materials often provide less support for teachers than the traditional materials with which they are familiar (Burkhardt, Fraser, & Ridgway, 1990), so this phase is especially important for curricula that are different than what teachers are used to. The goal of this phase is to ascertain if the supporting materials are flexible enough to support multiple situations, various modes of instruction, and different modes and styles of management. A new use of the Snapshot activity was created during this phase as we observed creative teachers placing large subitizing cards near the doorway to as children were lining up.

The next two phases involve summative research, with the goal of evaluating the effectiveness (e.g., in affecting teaching practices and ultimately
student learning) of the curriculum, now in its complete form, as it is implemented in realistic contexts. These two phases (9 and 10) differ from each other most markedly on the characteristic of scale.

**Phase 9. Summative Research: Small Scale**

This phase evaluates what can actually be achieved with typical teachers under realistic circumstances (Burkhardt et al., 1990) and may overlap in practice with phase 8. In multiple classrooms (2 to about 10), pre- and posttest (standardized instruments), experimental or quasi-experimental designs using measures of learning are often used, in conjunction with, and to complement, methodologies previously described. Relevant here is that the Building Blocks group not only outperformed the control group with large effects (0.85 SD for the number test), but that the largest of all the effects was on subitizing (1.56 SD, Clements & Sarama, 2007b).

**Phase 10. Summative Research: Large Scale**

With any curriculum, but especially one that differs from the familiar for teachers, evaluations must be conducted on a large scale (after considering issues of ethics and practical consequences, see Lester Jr. & Wiliam, 2002). Such research should use an embedded mixed methods design with as broad set of instruments to assess the impact of the implementation on participating children, teachers, program administrators, and parents, as well as document the fidelity of the implementation of the curriculum across diverse contexts. The objective should be to measure and analyze the critical variables, including contextual variables (e.g., settings, such as urban/suburban/rural; type of program; class size; teacher characteristics; student/family characteristics) and implementation variables (e.g., engagement in professional development opportunities; fidelity of implementation; leadership, such as principal leadership, as well as support and availability of resources, funds, and time; peer relations at the school; “convergent perspectives” of the developers, school administrators, and teachers in a cohort; and incentives used). A randomized experiment might be used to provide an assessment of the average impact of being exposed to the curriculum with embedded qualitative analyses.

We have created a model (TRIAD, for Technology-enhanced, research-based instruction, assessment, and professional development) and have evaluated it extensively with positive results, although effects decline longitudinally (Clements, Sarama, Spitler, Lange, & Wolfe, 2011; Sarama & Clements, 2013).

The TRIAD model holds that professional development should be ongoing, intentional, reflective, focused on mathematics content knowledge and children’s thinking and on learning trajectories, grounded in particular curriculum materials, situated in the classroom and the school. To realize this, we conducted both repeated (e.g., >10) full-day sessions of training in regular meetings and frequent coaching. Training included all three components of each learning trajectory, the goal, the developmental progression, and the
instructional activities and strategies (as in Fig. 3). To understand the goal, teachers study the mathematical content; examples for subitizing include the concept of cardinality and partitioning numbers. A key instructional use of learning trajectories is in formative assessment along the developmental progression. We worked with teachers to study the developmental progression for subitizing, analyze multiple video segments illustrating each level and discuss the mental “actions on objects” that constitute the defining cognitive components of each level; order tasks corresponding to those levels; and practice diagnosis in teams, with a couple of teachers exemplifying behaviors of children at different levels, and one teacher identifying the level of each (we used an online application; an update to it can be seen at learningtrajectories.org). Further, teachers need training in understanding, administering, and especially using data from new assessment strategies (Foorman, Santi, & Berger, 2007). TRIAD training focuses mainly on the curriculum-embedded assessment of Building Blocks’ Small Group Record Sheets.

Formative assessment requires more than identifying children’s levels of thinking. Teachers must select and modify instructional activities and strategies that are appropriate and effective for each level. To learn about instructional tasks and strategies, teachers practice the curriculum’s activities, but also analyze them to establish and justify their connection to a particular level of the developmental progression (as in Fig. 3).

Across all forms of professional development, the focus is on children’s thinking and learning. Conversations about children learning serves as way to address implementation issues. Although early mathematics is often an uncomfortable topic for early childhood educators, the newness of the learning trajectories for all participants helps establish a sense of shared learning and community. Each session in the last third of professional development includes scheduled time to discuss “learning stories” (Perry, Dockett, & Harley, 2007). Teachers show their record keeping on small group record sheets, and sometimes videos, and discuss their use of learning trajectories in teaching children, including challenges, questions, and successes. These discussions promote peer learning and collaboration and also motivate peers to solve implementation difficulties.

CONCLUSIONS AND FUTURE DIRECTIONS

We described our Curriculum Research Framework and illustrated its instantiation with the Building Blocks curriculum, focusing on the goal of subitizing. As diagrammed in Fig. 1, we believe the success it had stemmed from the complementary research bases, and especially the establishment of a set of cognitively grounded learning trajectories that contributed to all aspects of the projects. Future research should critically evaluate the veracity of these beliefs.

Future research and development might also evaluate the CRF’s implementation with other grade levels and other topics (see Doabler et al., 2014;
Kinzie et al., 2015). Just as important, our research designs could not identify which components of the CRF and TRIAD models and their instantiations are critical. Such research would be theoretically and practically useful.

The specific contribution of the learning trajectories per se needs to be disentangled and identified. In a present project, we are addressing this issue. In an IES-funded project entitled, “Evaluating the Efficacy of Learning Trajectories in Early Mathematics”, we are testing the efficacy of learning trajectories in a series of eight randomized clinical trials testing different aspects of LTs. These experiments will determine whether LTs are more efficacious than other approaches in supporting young children’s learning.

On a practical side, with funding from the Heising-Simons Foundation and the Bill and Melinda Gates Foundation, we are developing a technology-based tool for teachers and teacher trainers that extends a resource we created for the TRIAD evaluation. The Learning and Teaching with Learning Trajectories tool, or LT2, is a new, free resource for early mathematics (see www.LearningTrajectories.org). LT2 provides learning-trajectories-based math resources for teachers, caregivers, and parents. LT2 runs on all technological platforms, addresses new ages—birth to age 8 years—and including new alignments with standards and assessments, as well as new software for children from—their everyday activities, including art, stories, puzzles, and games.

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